Lagrangian Simulation of Dispersion in Turbulent Shear Flow with a Hybrid Computer

Two-dimensional turbulent diffusion is modeled with a generalized Langevin equation. Instantaneous velocities are simulated by using white noise filtered to simulate turbulence. By generating fluctuating signals that are partially correlated, it is possible to model diffusion for a turbulent shear field. The results show that the dispersion is independent of shear for long and short diffusion times but depend strongly on shear for intermediate times.

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SCOPE

Quantitative prediction of turbulent diffusion is important in a variety of industrial process applications and in the analysis of air pollution. Many attempts to predict turbulent diffusion have been reported. Most assume homogeneous, isotropic turbulence where there is no gradient of the mean velocity and zero shear stress (that is, zero cross correlation between the velocity fluctuations). However, in turbulent fields of practical importance, such as flow in conduits or in the atmospheric boundary layer, the results of this simple theory are not in accord with experiment. This suggests that the variation of the mean velocity with position and the existence of nonzero cross correlation of the turbulent velocity fluctuations are important to the diffusion process.

Corrsin (1953) extended these simple theories by including a constant gradient of the mean velocity in the Lagrangian diffusion equations written for a field of infinite extent having a point source. He neglected cross correlating terms, a situation which is physically unrealistic since the gradient can not exist in the absence of shear. In the limit of long diffusion times he showed that the presence of the gradient has a very large effect on the mean-square dispersion with no effect for small times. This still leaves the question: "How important is the existence of shear to the diffusion process?"

Recently, Bullin and Dukler (1974) proposed modeling the turbulent diffusion process by repeated solutions of the Langevin equation on a hybrid computer. In this approach, velocity fluctuations are simulated by using white noise filtered in such a way as to reproduce the spectrum and the root-mean-square value of the turbulent fluctuations. Sequences of such fluctuations were integrated to track the position of the fluid particle. After many such solutions, the statistics of the distribution of fluid particles (mean-square dispersion or concentration) can be calculated. It was shown that such an approach is simple in its software and economical of machine time with a modern hybrid computer.

In this work, the method of Bullin and Dukler is extended to include the presence of turbulent shear stress. This is done by generating input fluctuating signals which are correlated with each other to any degree desired. The technique is based on the principle of signal modulation. Further, the Corrsin's theoretical approach is extended to include the case of perfect correlation between cross velocities. In this way the validity of the earlier Corrsin theory can be tested against the simulation, and the effect of the presence of shear on turbulent dispersion can be evaluated.

CONCLUSIONS AND SIGNIFICANCE

Results of the simulation showed that turbulent diffusion from a point source in a two-dimensional field of infinite extension with a uniform velocity gradient and with zero shear is characterized by the following relations in the limits of long and short diffusion times (Figure 5), where subscript 1 designates the flow direction.

For long diffusion time

$$\sqrt{\overline{y_1^2}} \propto t^{3/2}$$

$$\sqrt{\overline{y_2^2}} \propto t^{1/2}$$

For short diffusion time

$$\sqrt{\overline{y_1^2}} \simeq (V_1)_{\rm rms} t$$

$$\sqrt{\overline{y_2^2}} \simeq (V_2)_{\rm rms} t$$

These results agree with those predicted by Corrsin (1953). Numerical solution of the Corrsin equation shows excellent agreement between his theory and this simulation procedure, thus giving considerable credence to the Corrsin prediction and the present simulation.

In the presence of both gradient and shear, dispersion is shown to be independent of the degree of correlation between velocities for long and short diffusion times. However, at all intermediate times the mean-square dispersion is significantly influenced by the existence of cross correlation (Figure 6). Shear is shown to lower the degree of dispersion. The effect depends on the degree of

correlation. At intermediate diffusion times, the contribution due to shear depends strongly on the magnitude of the velocity gradient.

This new method of simulating dispersion with shear and nonuniform mean velocity will permit the prediction of turbulent diffusion in boundary layers and conduits.

Taylor (1921) introduced the Lagrangian statistical approach for analyzing dispersion in homogeneous, isotropic turbulence, and his concept has been amplified and extended by many others. However, very few theoretical treatments of diffusion for inhomogeneous turbulent shear flow are available (Batchelor and Townsend, 1956; Corrsin, 1950; Hinze, 1959). Similarly, there are few experimental studies which have focused on the effects of shear (Corrsin and Uberoi, 1951; Hinze and van der Hegge Zijnen, 1951; Skramstad and Schubauer, 1938). Even these theoretical and experimental studies consider diffusion times so small that they provide information only for the initial stages of diffusion.

Corrsin (1953) explored the effect of a gradient of the mean velocity on turbulent diffusion by considering a hypothetical field of infinite extent with a uniform velocity gradient but with the correlation between cross velocity components being zero. His main results may be summarized as follows. In two-dimensional diffusion from a point source the position of a fluid particle at time t is

$$y_1(t) = \int_o^t \left[\frac{d\bar{U}}{dx_2} y_2(t_1) + v_1(t_1) \right] dt_1 \qquad (1)$$

$$y_2(t) = \int_o^t v_2(t_1) dt_1 \qquad (2)$$

where y and v represent the Lagrangian position and velocity, while x and u are the corresponding Eulerian quantities. Assuming that the autocorrelation of the velocity component becomes zero at large values of time, he concluded that the dispersion is characterized by the following relationships:

For long diffusion time

$$\sqrt{\overline{y_1^2}} \propto t^{3/2} \qquad (3)$$

$$\sqrt{\overline{y_2^2}} \propto t^{1/2} \qquad (4)$$

For short diffusion time

$$\sqrt{\overline{y_1^2}} \simeq (v_1)_{\text{rms}} t \qquad (5)$$

$$\sqrt{\overline{y_2^2}} \simeq (v_2)_{\text{rms}} t \qquad (6)$$

He also presented a qualitative discussion of the effect of shear. Since the presence of a gradient in the absence of correlation is physically unrealistic, the only results of interest were the limiting conditions of long and short diffusion times. While shown to be true for diffusion in an isotropic field, Hinze (1961) questioned the validity of the assumption of zero autocorrelation at long diffusion times in the presence of a velocity gradient.

Riley and Corrsin (1971) attempted to simulate the dispersion for homogeneous turbulent shear flows. The method involved setting up a mathematical model of a random Eulerian field and computing the dispersion in this field by using a digital computer. Their results were in good agreement with the long dispersion time prediction of Corrsin's approach discussed above and in fair agreement with the predictions for short diffusion times. The Eulerian field they selected produced an unrealistic relationship between the Lagrangian and Eulerian velocity correlation. Therefore, the use of these results to confirm Corrsin's theoretical results is in doubt.

In this paper the process of turbulent diffusion is modeled by using a hybrid computer. The Lagrangian turbulent field is simulated by using band limited white noise. Methods are presented by which the two fluctuating quantities can be correlated to any desired degree, and uniform or position varying velocity can be imposed. In this way it is possible to test Corrsin's (1953) theoretical results for constant velocity gradients in the absence of shear at long and short diffusion times, to extend the result to all intermediate times, and to explore the effect of correlation between the velocities on the process of diffusion.

Borgman (1969) attempted to simulate correlated random processes using a digital filter. Hoshiya and Tieleman (1971) and Shinozuka and Jan (1971) explored trigonometric methods, while Sinha (1973) developed a trigonometric method combined with the application of a fast Fourier transform. Those approaches make use of a digital computer. Though Sinha's methods are more efficient in computer time than the others mentioned above, they are still complex with respect to software and require much machine time.

HYBRID COMPUTER SIMULATION

Dispersion Simulation Equation

Simulation of turbulent diffusion on a modern hybrid computer was recently explored by Bullin and Dukler (1974). In this approach the position of the fluid particle was designated as a stationary Markovian process. The instantaneous Eulerian velocity was simulated with band limited white noise, the random signals in the three coordinate directions being uncorrelated. The use of band limited white noise is justified by the fact that the diffusion process is controlled by the low frequency end of the process, and the portion of the velocity spectrum is essentially uniform. The assumption of a constant time scale ratio was used to relate Eulerian and Lagrangian signals. Comparison of predicted and measured concentration distributions in boundary layers and in the atmosphere was very satisfactory.

Further theoretical analysis now suggests that a sounder representation of the Markov process can be obtained by considering the instantaneous velocity as the stochastic process. In this way it is possible to generate turbulent velocities from white noise driving functions having spectra very similar to those observed from measurements over the full frequency range. Details of this analysis will appear in a future publication. Although this new theory rests on a sounder basis, the agreement between theory and experiment is not significantly different from that obtained by Bullin and Dukler (1974). Thus, the Bullin-Dukler model will be used in the method discussed below to test Corrsin's speculation on the effect on dispersion of a uniform velocity gradient and to determine the influence of correlation between velocities on the dispersion.

In order to simulate a two-dimensional turbulent field with a gradient of the mean velocity as analyzed by Corrsin, the following model for instantaneous displacement position is used:

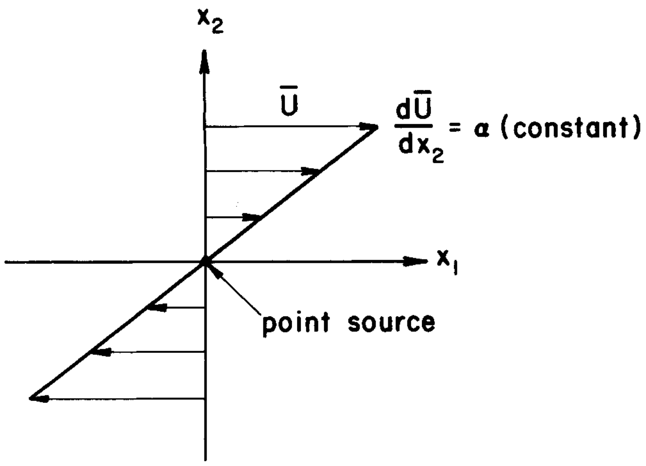

Fig. 1. The flow field.

$$\frac{dy_1}{dt} = n_1(t) \qquad y_1(0) = 0 \tag{7}$$

$$\frac{dy_2}{dt} = n_2(t) \qquad y_2(0) = 0 \tag{8}$$

where n_1 and n_2 are band limited white noise.

Figure 1 shows a flow field having a point source diffusion with a constant gradient of the mean velocity α in an unbounded turbulent field. Figure 2 is a schematic diagram of the analogue circuit which simulates this system. If the two input signals n_1 and n_2 are uncorrelated, the circuit simulates the simplified model of dispersion with a gradient of the mean velocity in an otherwise isotropic field as explored by Corrsin (1953). When the input signals are correlated to any degree, the influence of turbulent shear is included.

Equations (7) and (8) are particularly simple forms of the Langevin equation and can be solved rapidly and repeatedly on the analogue component of a hybrid computer for a sequence of realizations of the random forcing functions n_1 and n_2. The resulting values of y_1 and y_2 for each realization are digitized and stored digitally. The digital computer is used to execute the bookkeeping which keeps track of the locations of the fluid packets and calculates the sample mean of the squared dispersion distance for each diffusion time t.

Hardware

The hybrid computer at the University of Houston consists of an IBM 360/44 digital computer, Hybrid Systems Model SS100 analogue computer, and a Hybrid Systems Model 1044 linkage. The analogue computer is capable of repeated solution at a frequency of over 1 000 times per second, and the components are designed for bandwidth of over 100 KHz. Thus, several minutes of simulations time provides over 100 000 realizations of the dispersion process with high accuracy in the resulting averages of mean-square dispersion.

Generation of Forcing Functions

To implement a Langevin equation on the Hybrid computer, forcing signals are required which have the required statistical characteristics and which have the desired degree of cross correlation. The two-dimensional case will be considered here. The forcing functions are assumed to be band limited white noise which have been shown by Bullin and Dukler (1974) to represent those aspects of the spectral character of turbulence important to the diffusion process.

The basis of the present method for providing correlation between two signals rests in the principle of linear

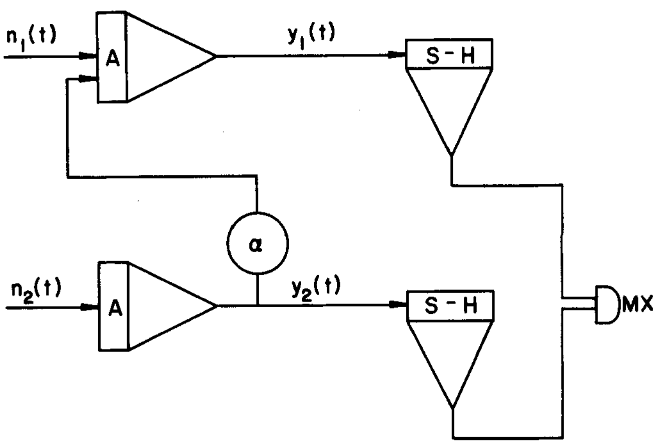

Fig. 2. Analogue circuit for simulation.

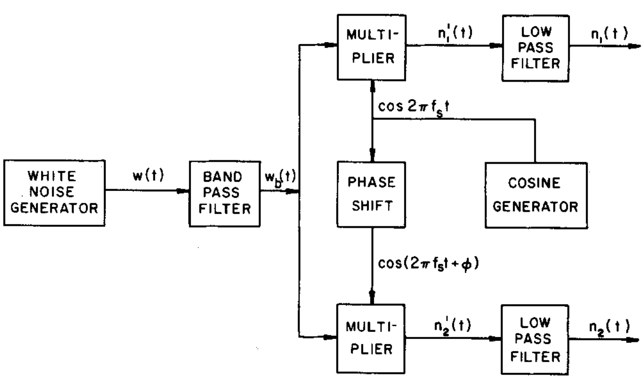

Fig. 3. Analogue generation of two signals with varying degrees of correlation.

modulation widely used in the field of communication. The general scheme is shown in Figure 3. A detailed explanation of the principle is given in Appendix A. The white noise was generated by a Hewlett Packard's Precision Noise Generator Model 8057A. A Krohn-Hite Band Pass Filter Model 330 AR and a Hewlett Packard's Low Pass Filter Model 5489A were used as filters. The required cosine function was obtained from a Wavetek Model 115 (Phase Lock/Trigger VCG), and the same generator provided a mean for controlling the phase angle shift.

For purposes of this simulation, root-mean-square values of n_1 and n_2 were arbitrarily selected at 950 m/hr. Each signal was essentially white noise band limited with a cutoff frequency at about 10 cycles/m. The normalized autocorrelation function which resulted appears in Figure 4. This function is closely fit by the equation

$$R(\tau) = e^{-5\tau} \frac{\sin(30\pi\tau)}{30\pi\tau} \tag{9}$$

as shown by the points in Figure 4.

THEORY

General equations for diffusion in a two-dimensional field with a constant gradient of mean velocity can be derived starting with the integral equation for displacement, Equations (1) and (2). The mean-square dispersion in the two directions then becomes

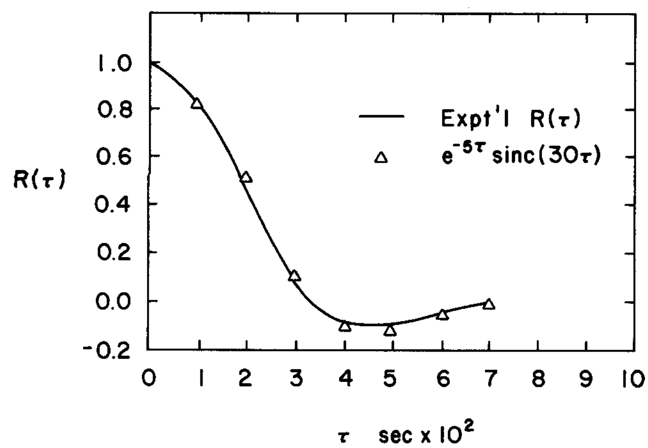

Fig. 4. Autocorrelation of random signal.

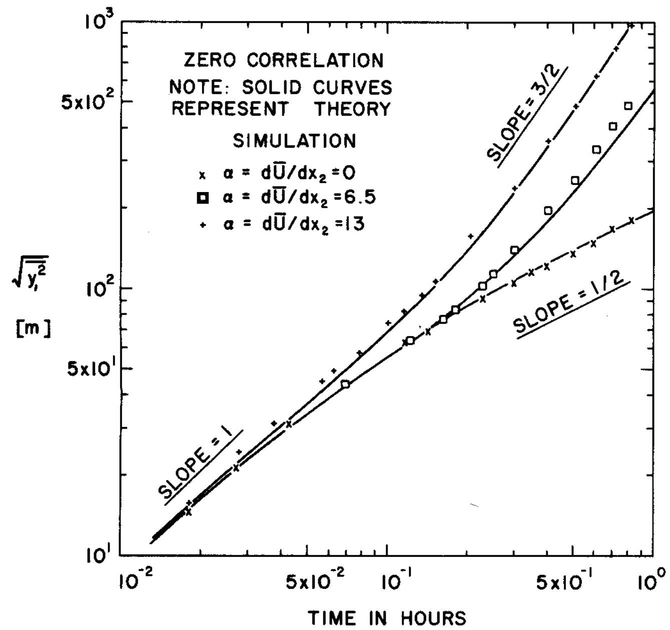

Fig. 5. Comparison of theory and simulation: effect of velocity gradient with zero correlation.

$$\overline{y_1^2(t)} = \int_o^t \int_o^t \overline{v_1(t_1)v_1(t_2)}\,dt_1dt_2$$

$$+ \alpha^2 \int_o^t \int_o^t \overline{y_2(t_1)y_2(t_2)}\,dt_1dt_2$$

$$+ 2\alpha \int_o^t \int_o^t \overline{y_2(t_2)v_1(t_1)}\,dt_1dt_2 \quad (10)$$

$$\overline{y_2^2(t)} = \int_o^t \int_o^t \overline{v_2(t_1)v_2(t_2)}\,dt_1dt_2 \quad (11)$$

The first integral in each of the above equations measures the dispersion which would be expected in an isotropic field. The second term in the y_1 equation is the contribution due to the uniform velocity gradient α in the absence of correlation between the velocity components, and the last term is the contribution resulting from the cross correlation. It is necessary to relate the double integrals to single integrals of the relevant correlations. The equation for zero cross correlation has been given by Hinze (1961), but it contains an error. The development for the last two terms of the y_1 equation appear in Appendix B. The first term in both equations can be reduced to a single integral by the method given first by Kampé de Fériet (1939). The results are

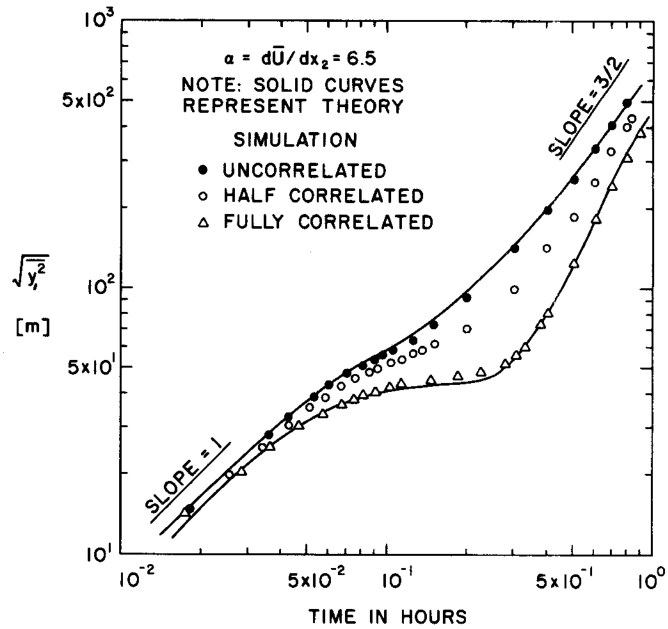

Fig. 6. Comparison of theory with simulation: variation of correlation with $\frac{d\bar{U}}{dx_2} = 6.5$.

$$\overline{y_1^2(t)} = 2(v_1)^2_{rms} \int_o^t (t-\tau)R_1(\tau)d\tau$$

$$+ \alpha^2(v_2)^2_{rms} \int_o^t \left[\frac{2}{3}t^3 - t^2\tau + \frac{1}{3}\tau^3 \right] R_2(\tau)d\tau$$

$$+ 2\alpha(v_1)_{rms}(v_2)_{rms} \int_o^t \int_o^t (t-t_2)R_{12}(t_1,t_2)dt_1dt_2 \quad (12)$$

$$\overline{y_2^2(t)} = 2(v_2)^2_{rms} \int_o^t (t-\tau)R_2(\tau)d\tau \quad (13)$$

For the particular case where v_1 and v_2 are perfectly correlated, $v_1(t) = -v_2(t)$, and then the last term in Equation (12) becomes

$$-2\alpha(v_1)^2_{rms} \int_o^t (t^2 - t\tau)R_1(\tau)d\tau \quad (14)$$

RESULTS

Constant Gradient of Mean Velocity and $R_{12} = 0$

This case was simulated on the hybrid computer with the two input signals uncorrelated, with each signal having the normalized correlation shown in Figure 4. Each value of $\overline{y^2}$ was obtained from an average of 150 000 realizations to insure very high statistical reliability. The results are shown as data points in Figure 5 for a series of values of the mean velocity gradient α. The results of this simulation were compared with the theory by numerically solving Equation (12) with the last term zero, by using the equation for $R(\tau)$

$$R(\tau) = e^{-5\tau} \frac{\sin(30\pi\tau)}{30\pi\tau}$$

which, as shown in Figure 4, is an excellent fit to the correlation. The agreement is shown to be very good indeed. Thus, the hybrid computer method of simulation agrees precisely with the theory of Corrsin (1953) in the limits of small and large diffusion times as well as for all intermediate times. As Corrsin speculated, for

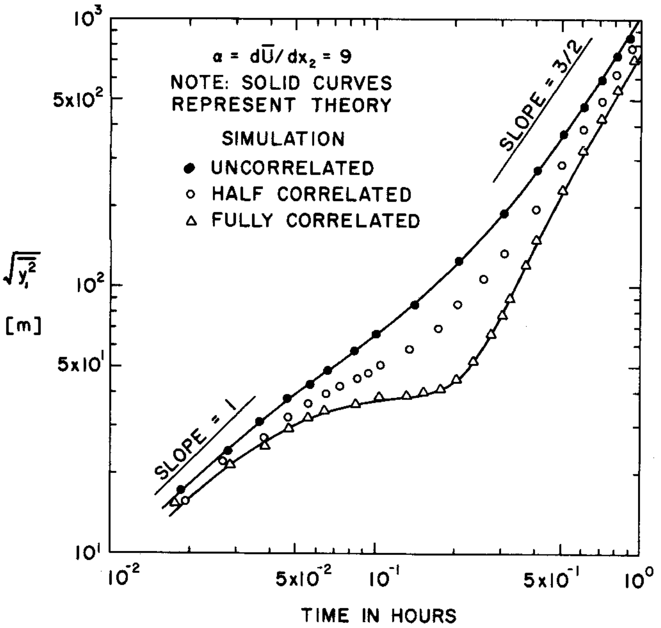

Fig. 7. Comparison of theory with simulation: variation of correlation with $\frac{d\bar{U}}{dx_2} = 9$.

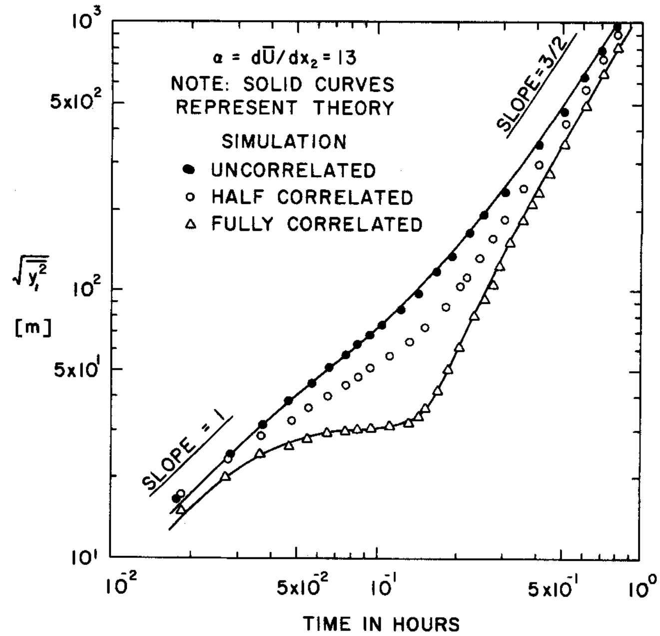

Fig. 8. Comparison of theory with simulation: variation of correlation with $\frac{d\bar{U}}{dx_2} = 13$.

short diffusion times, when a constant gradient of mean velocity is imposed, the time dependence of the root-mean-square dispersion is as the first power, the same as observed in the absence of the gradient. But at long time of diffusion, this dependence changes from the one half power in the absence of a gradient to the 1.5 power when $\alpha \neq 0$. These results give validity both to the method of simulation and to the Corrsin results because of the very precise agreement of the results from two very different methods.

Constant Gradient of the Mean Velocity and $R_{12} \neq 0$

With the above result, it is now possible to explore with confidence the effect of the cross correlation of the velocity components on the dispersion. Figures 6 to 8 show the results of the simulation for a series of values of α and various degrees of cross correlation from 0 to 1.0 [see Appendix A for definition]. The case of perfect correlation can be calculated from the extensions of the Corrsin analysis as indicated in Equations (12) and (14). Again, the excellent agreement between the theoretical calculations and the simulation is evidence of the validity of this very efficient simulation technique. The presence of cross correlation (that is, turbulent shear) lowers the degree of dispersion in the flow direction. However, in the limits of sufficiently long or short diffusion time, the results agree with those for zero shear. The intermediate time interval over which the results diverge from the case of no shear depends strongly on the gradient of the mean velocity.

Figure 9 shows the shape of the isoprobability contours for four cases of interest. The first three were explored by Corrsin (1953), and the results of these simulations agree with his conclusions. Concentric contours are expected in the isotropic case in the absence of shear and velocity as shown in sketch A. When $\alpha \neq 0$, but in the absence of shear, the contours are nearly circular near the origin but elongate as distance is increased from the origin, the axis of the ellipse shifting toward the x_1 axis with increasing time (see sketch C).

Sketch B shows the results in the absence of mean velocity but with a uniform turbulent shear. This is, of

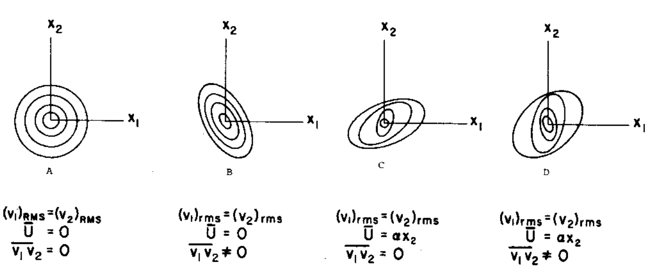

Fig. 9. Isoprobability contours.

course, as physically unrealistic a situation as is the case of a gradient of the mean velocity in the absence of shear. Sketch D shows the contours for the physically most acceptable case where $\alpha \neq 0$ and where turbulent shear exists. The contours are elongated near the origin, with the axis having a negative slope. As time increases, the slope of the axis moves to positive values approaching the x_1 axis for long times.

CONCLUSIONS

A method for simulation of turbulent dispersion on a modern hybrid computer is presented which is simple to program, very rapid to execute, and gives results of high statistical reliability. Mean-square dispersions were obtained by simulating a two-dimensional field having a linear distribution of the mean velocity, the Lagrangian fluctuating velocities being uncorrelated. The results agree precisely with a theory based on a speculation of Corrsin (1953) in the limits of short and long diffusion times as well as with an analytical solution of the Corrsin equations for intermediate times. This result suggests both the validity of the simulation method as well as of the Corrsin model.

A method is introduced for simulation of diffusion with two fluctuating signals having any desired degree of correlation. Simulation of a flow field with linear velocity distribution and various degree of correlation between velocities shows that in the limit of long and short diffusion times, the dispersion is independent of the degree of correlation. However, the mean-square dispersion is strongly influenced by the degree of correlation at all intermediate times.

This result makes possible the use of this method to simulate diffusion in turbulent boundary layers and conduit flow, where gradients of the mean velocity *and* the correlation between velocities vary with position.

NOTATION

$F(t)$ = defined as $\int_o^t y(t')dt'$
f_m = maximum frequency of a forcing function
f_s = frequency of a carrier signal
f_c = cutoff frequency of a noise
f_o = fundamental frequency
$n(t)$ = forcing function in Langevin equation
$R(\tau)$ = autocorrelation of velocity fluctuation
$R_{12}(t_1, t_2)$ = cross correlation between fluctuating velocity components
$S(f)$ = power spectrum
t = time
$\overline{U}$ = mean velocity
v = fluctuating velocity
v_{rms} = root-mean-square of fluctuating velocity
$w(t)$ = white noise
$w_b(t)$ = band passed white noise
x = Eulerian coordinate
y = Lagrangian coordinate
α = mean velocity gradient
θ = phase angle
ϕ = phase angle

Subscripts

1 = coordinate in direction of mean flow
2 = coordinate which is perpendicular to the mean flow

Superscripts

– = mean value

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APPENDIX A: THE GENERATION OF TWO SIGNALS WITH VARYING DEGREES OF CORRELATION

The objective is to generate an ideal band limited white noise with maximum frequency f_m, whose degree of correlation with a second such signal can be regulated. The process is shown in Figure A1 and is described as follows:

1. White noise having a cutoff frequency of f_c is processed through a band pass filter with bandwidth $2f_m$ and center frequency f_s.
2. The output signal is multiplied by the function $2\cos(2\pi f_s t)$, where f_s is selected to be greater than $2f_m$.
3. The resulting signal is processed through a low pass filter with cutoff frequency f_m.

The output of the final low pass filter is a time varying signal $n_1(t)$. Now, consider a system for generating two such signals, the first carrying a cosine wave of frequency $2\pi f_s$ and the second carrying a cosine where phase is shifted by θ, $\cos(2\pi f_s t + \theta)$. This system is shown in Figure 3. The two signals $n_1(t)$ and $n_2(t)$ are correlated to a degree depending on the phase shift, θ. This is shown below. $W_b(t)$ may be expressed as a Fourier series:

$$W_b(t) = \sum_{n=0}^{\infty} a_n \cos(2\pi n f_o t + \theta_n) \tag{A1}$$

Then, the two signals $n_1'(t)$ and $n_2'(t)$ are given as

$$n_1'(t) = 2 \sum_{n=0}^{\infty} a_n \cos(2\pi n f_o t + \theta_n) \cos(2\pi f_s t + \phi_1) \tag{A2}$$

$$n_2'(t) = 2 \sum_{n=0}^{\infty} a_n \cos(2\pi n f_o t + \theta_n) \cos(2\pi f_s t + \phi_2) \tag{A3}$$

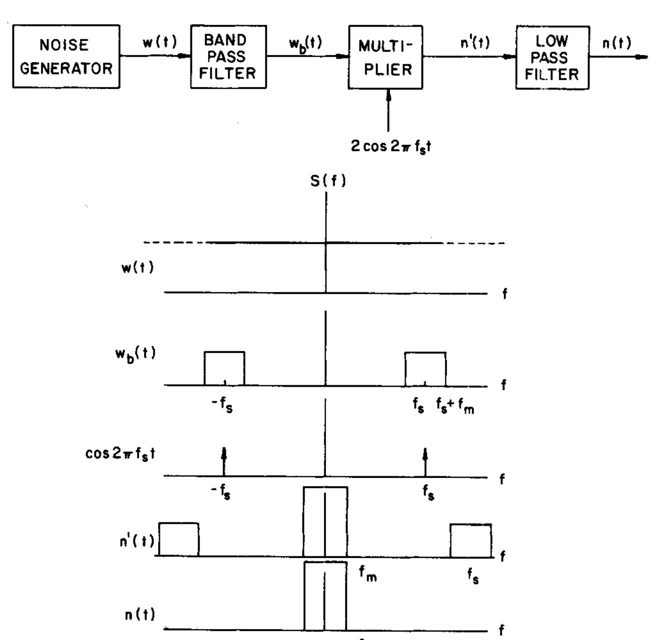

Fig. A-1. Processing of a single signal.

Equation (A2) may be simplified:

$$n_1'(t) = \sum_{n=0}^{\infty} a_n \{\cos[2\pi(f_s + nf_o)t + \theta_n + \phi_1] + \cos[2\pi(f_s - nf_o)t + \phi_1 - \theta_n]\} \quad (A4)$$

Thus, if the signal $n_1'(t)$ is passed through a low pass filter whose cutoff frequency is less than $f_s/2$, the resultant signal $n_1(t)$ is expressed as

$$n_1(t) = \sum_{n=0}^{N} a_n \{\cos 2\pi(f_s - nf_o)t + \phi_1 - \theta_n\} = \sum_{m_1=0}^{m} a_{m1} \cos(2\pi m_1 f_o t + \phi_{m1}) \quad (A5)$$

Similarly, $n_2(t)$ becomes

$$n_2(t) = \sum_{m_2=0}^{m} a_{m2} \cos(2\pi_{m2} f_o t + \phi_{m2}) \quad (A6)$$

where $\phi_{m1} = \phi_1 - \theta n$, and $\phi_{m2} = \phi_2 - \theta n$. In accordance with Equations (A5) and (A6), $n_1(t)$ and $n_2(t)$ are simply phase shifted with respect to each other.

For simplicity, the two signals will be rewritten as

$$n_1(t) = \sum_{m_1=0}^{m} a_{m1} \cos 2\pi m_1 f_o t \quad (A7)$$

$$n_2(t) = \sum_{m_2=0}^{m} a_{m2} \cos(2\pi m_2 f_o t + \phi) \quad (A8)$$

where $\phi = \phi_{m1} - \phi_{m2}$. $n_2(t)$ may be rewritten as

$$n_2(t) = \sum_{m_2=0}^{m} a_{m2} [(\cos 2\pi m_2 f_o t)\cos\phi - (\sin 2\pi m_2 f_o t)\sin\phi] = n_1(t)\cos\phi - \sum_{m_2=0}^{m} a_{m2} (\sin 2\pi m_2 f_o t)\sin\phi \quad (A9)$$

Therefore, from Equations (A7) and (A9), the cross correlation of the two signals is given as

$$\overline{n_1(t_1)n_2(t_2)} = \overline{n_1(t_1)n_1(t_1)}\cos\phi - \sin\phi \overline{\left[\sum_{m_1=0}^{m} a_{m1} \cos 2\pi m_1 f_o t_1\right]\left[\sum_{m_2=0}^{m} a_{m2} \sin 2\pi m_2 f_o t_2\right]} \quad (A10)$$

Since

$$\overline{n_1(t_1)n_1(t_2)} = R_{n1}(t)$$

where $\tau = t_2 - t_1$, and $\overline{\cos 2\pi m_1 f_o t_1 \sin 2\pi m_2 f_o t_2} = 0$ for any m_1 and m_2. Equation (A10) is simplified to give

$$R_{12}(t_1, t_2) = \overline{n_1(t_1)n_2(t_2)} = R_{n1}(\tau)\cos\phi \quad (A11)$$

In accordance with Equation (A11), if ϕ is 90 deg., then $\overline{n_1(t_1)n_2(t_2)} = 0$, which implies $n_1(t)$ and $n_2(t)$ are statistically independent of each other. Further, the degree of correlation may be controlled by simply adjusting the phase shift. In this work, shift angles of 90, 150, and 180 deg. were used and designate uncorrelated, half correlated, and fully correlated signals.

APPENDIX B: DEVELOPMENT OF EQUATION (12) FROM EQUATION (10)

The Second Term in Equation (10)

For the purpose of simplifying the second term in Equation (10), define $F(t)$ as

$$F(t) = \int_o^t y(t')dt' = \int_o^t \int_o^{t'} v(t'')dt''dt' \quad (B1)$$

Then the problem is to determine $\overline{F(t)^2}$. The following relationships are obtained from the rules of differentiation and the fact that $dy(t)/dt = v(t)$:

$$\frac{d^2F^2(t)}{dt^2} = 2y(t)^2 + 2F(t)\,v(t) \quad (B2)$$

$$\frac{d^2\overline{F^2(t)}}{dt^2} = 2(\overline{y(t)^2} + \overline{F(t)\,v(t)}) \quad (B3)$$

Using partial integration of Equation (B1), we get

$$F(t) = \int_o^t (t - t')\,v(t')dt' \quad (B4)$$

With Equations (B4), the mean of the product of $F(t)$ and $v(t)$ is

$$\overline{v(t)F(t)} = v^2_{\text{rms}} \int_o^t \tau R_v(\tau)d\tau \quad (B5)$$

where a change of variable was made. The result of Kampé de Fériet (1939) and Equation (B5) may be used in Equation (B3) to give

$$\frac{\overline{d^2F^2(t)}}{dt^2} = v^2_{\text{rms}} \int_o^t (4t - 2\tau)R_v(\tau)d\tau \quad (B6)$$

Integration of Equation (B6) with the initial condition of $y(0) = 0$ gives

$$\overline{F^2(t)} = v^2_{\text{rms}} \int_o^t \left(\frac{2}{3}t^3 - t^2\tau + \frac{1}{3}\tau^3\right) R_v(\tau)d\tau \quad (B7)$$

So the second term in Equation (10) may be rewritten as

$$\alpha^2(v_2)^2_{\text{rms}} \int_o^t \left(\frac{2}{3}t^3 - t^2\tau + \frac{1}{3}\tau^3\right) R_2(\tau)d\tau \quad (B8)$$

and this is the second term in Equation (12).

The Third Term in Equation (10)

Since the third term is the mean of the product of the two integrations, it may be conveniently represented as

$$\overline{\int_o^t v_1(t')dt' \int_o^t y_2(t'')dt''}$$

This term may be simplified by using partial integration as

$$\overline{\int_o^t v_1(t')dt' \int_o^t (t - t'')v_2(t'')dt''} = \int_o^t \int_o^t (t - t'')\overline{v_1(t')v_2(t'')}dt'dt'' \quad (B9)$$

Equation (B9) gives the relationship

$$2\alpha \int_o^t \int_o^t \overline{y_2(t_2)v_1(t_1)}\, dt_1dt_2 = 2\alpha(v_1)_{\text{rms}}(v_2)_{\text{rms}} \int_o^t \int_o^t (t - t_2)R_{12}(t_1, t_2)dt_1dt_2 \quad (B10)$$

where $R_{12}(t_1, t_2)$ is defined as

$$R_{12}(t_1, t_2) = \frac{\overline{v_1(t_1)v_2(t_2)}}{(v_1)_{\text{rms}}(v_2)_{\text{rms}}} \quad (B11)$$

The right side of Equation (B10) is the third term in Equation (12).

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